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QUASIDIAGONALITY AND THE HYPERINVARIANT SUBSPACE PROBLEM

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ABSTRACT

In a sequence of recent papers, [11], [13], [9] and [5], the authors (together with H. Bercovici and C. Foias) reduced the hyperinvariant subspace problem for operators on Hilbert space to the question whether every C_{00}-(BCP)-contraction that is quasidiagonal and has spectrum the unit disc has a nontrivial hyperinvariant subspace (n.h.s.). An essential ingredient in this reduction was the introduction of two new equivalence relations, ampliation quasisimilarity and hyperquasisimilarity, defined below. This note discusses the question whether, by use of these relations, a further reduction of the hyperinvariant subspace problem to the much-studied class $(\mathbf{N}+\mathbf{K})$ (defined below) might be possible.

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1. Preliminaries

In this note a fixed separable, infinite dimensional, complex Hilbert space is denoted by $\mathcal{H}$ and the algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. For T in $\mathcal{L}(\mathcal{H})$ we write, as usual, $\sigma(T)$, $\sigma_e(T)$, and $\sigma_{le}(T)$, for the spectrum, essential (Calkin) spectrum, and left essential spectrum of T, respectively. The set of all scalar multiples of $1_{\mathcal{H}}$ is written as $\mathbb{C}1_{\mathcal{H}}$, the closed ideal of all compact operators in $\mathcal{L}(\mathcal{H})$ by $\mathbf{K}$ or $\mathbf{K}(\mathcal{H})$, and the Calkin map of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H})/\mathbf{K}$ by π. If $\mathcal{S}$ is any subset of $\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{S}'$ the **commutant** of S, i.e., $\mathcal{S}' = \{T \in \mathcal{L}(\mathcal{H}) : ST = TS$ for every S in $\mathcal{S}\}$. Recall that a subspace (closed linear manifold) $\mathcal{M} \subset \mathcal{H}$ is said to be a **nontrivial hyperinvariant subspace** (n.h.s.) for some T in $\mathcal{L}(\mathcal{H})$ if $(0) \neq \mathcal{M} \neq \mathcal{H}$ and $S\mathcal{M} \subset \mathcal{M}$ for each S in $\{T\}'$, and that the (open) **hyperinvariant subspace problem** (for operators on Hilbert space) is the question whether every operator T in $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ has a n.h.s.

With $\mathbb{N}$ the set of positive integers and $\mathbb{D}$ the open unit disc in $\mathbb{C}$, recall also that a completely nonunitary (c.n.u.) contraction T in $\mathcal{L}(\mathcal{H})$ is called a (BCP)-operator if $\mathbb{D} \cap \sigma_e(T)$ is a dominating set (as defined in [6]) for the unit circle $\mathbb{T} := \partial\mathbb{D}$, and that the class $C_{00}(\mathcal{H})$ consists of the set of all (c.n.u.) contractions T in $\mathcal{L}(\mathcal{H})$ such that both sequences $\{T^n\}_{n\in\mathbb{N}}$ and $\{T^{*n}\}_{n\in\mathbb{N}}$ converge to zero in the strong operator topology (SOT). The class of (BCP)-operators, introduced in [7], played an important role in the highly successful theory of dual algebras of operators, and is a subset of the larger class $\mathbb{A}_{\aleph_0}$ (see, e.g., [4] for more information about the theory of dual algebras). It is well-known that operators in $\mathbb{A}_{\aleph_0}$ have several good properties. For instance, every direct sum of strict contractions can be realized, up to unitary equivalence, as a compression to some semi-invariant subspace of an arbitrary operator in $\mathbb{A}_{\aleph_0}$ [3]. Moreover, the lattice $\mathrm{Lat}(T)$ of invariant subspaces of any operator T in $\mathbb{A}_{\aleph_0}$ is known to be so large that it contains a sublattice isomorphic to the lattice of all subspaces of $\mathcal{H}$ [3, Theorem 4.8], and such a lattice also contains a countably infinite family $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ of cyclic invariant subspaces with the property that $\mathcal{M}_m \cap \mathcal{M}_n = (0)$ whenever $m \neq n$ [2].

Also recall that operators S and T in $\mathcal{L}(\mathcal{H})$ are called **quasisimilar** (notation: $S \sim T$) if there exist quasiaffinities X and Y in $\mathcal{L}(\mathcal{H})$ (i.e., $\ker X = \ker X^* = \ker Y = \ker Y^* = (0)$) such that $SX = XT$ and $YS = TY$. For any cardinal number n satisfying $1 \leq n \leq \aleph_0$, we denote by $\mathcal{H}^{(n)}$ the direct sum of n copies of $\mathcal{H}$ (with $\mathcal{H}^{(\aleph_0)}$ indexed by the ordinal number ω), and by $T^{(n)}$ the direct sum (ampliation) of n copies of T acting on $\mathcal{H}^{(n)}$ in the obvious fashion. Following [11] we say that S and T are **ampliation quasisimilar** (notation: $S \overset{a}{\sim} T$) if there exist cardinal numbers $1 \leq n_1, n_2 \leq \aleph_0$ such that $S^{(n_1)} \sim T^{(n_2)}$ (or, equivalently, if $S^{(\aleph_0)} \sim T^{(\aleph_0)}$). It was shown in [11] that $\overset{a}{\sim}$ is an equivalence relation on $\mathcal{L}(\mathcal{H})$ weaker than quasisimilarity and that if $S \overset{a}{\sim} T$, then S has n.h.s. if and only if T does. Moreover, in [9] the equivalence relation of **hyperquasisimilarity** (notation: $\overset{h}{\sim}$) on $\mathcal{L}(\mathcal{H})$ was introduced, stronger than quasisimilarity but weaker than similarity, and having the property that if $S \overset{h}{\sim} T$, then the hyperinvariant subspace lattices $\mathrm{Hlat}(S)$ and $\mathrm{Hlat}(T)$ are lattice isomorphic (notation: $\mathrm{Hlat}(S) \equiv \mathrm{Hlat}(T)$). For more information about $\overset{h}{\sim}$, see [9]. In particular, for our purposes below, it suffices to know that if $S = \bigoplus_{n\in\mathbb{N}} S_n$, $T = \bigoplus_{n\in\mathbb{N}} T_n$, and S_n is similar to T_n for every $n \in \mathbb{N}$, then $S \overset{h}{\sim} T$ and consequently $\mathrm{Hlat}(S) \equiv \mathrm{Hlat}(T)$.

Furthermore, recall from [12] that an operator T in $\mathcal{L}(\mathcal{H})$ is **quasidiagonal** (notation: $T \in (QD)(\mathcal{H})$) if there exists an increasing sequence $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ of finite rank projections converging in the SOT to $1_{\mathcal{H}}$ and satisfying $\|P_nT - TP_n\| \to 0$, and T is **block diagonal** (notation: $T \in (BD)(\mathcal{H})$) if T is unitarily equivalent to a countably infinite (orthogonal) direct sum of operators each of which acts on a finite dimensional space. The basic structure theorem from [12] concerning (QD) is that $T \in (QD)(\mathcal{H})$ if and only if for every $\varepsilon > 0$ there exist $B_\varepsilon \in (BD)(\mathcal{H})$ and $K_\varepsilon \in \mathbf{K}(\mathcal{H})$ such that $T = B_\varepsilon + K_\varepsilon$ and $\|K_\varepsilon\| < \varepsilon$. Finally, we write $(A) \subset \mathcal{L}(\mathcal{H})$ for the collection of all **algebraic** operators, i.e., the set of $T \in \mathcal{L}(\mathcal{H})$ such that $p(T) = 0$ for some nonzero polynomial p.

2. Reductions

With the notation and terminology introduced above, we can state now two of the reductions of the hyperinvariant subspace problem obtained in the sequence of four papers mentioned above. The new equivalence relations $\overset{a}{\sim}$ and $\overset{h}{\sim}$ were used, together with the theory of closed similarity orbits of operators, to obtain the following.

THEOREM 2.1: a) *If every C_{00}-(BCP)-contraction $T \in (QD)(\mathcal{H})$ such that $\sigma_e(T) = \mathbb{D}^-$ has a n.h.s., then every operator in $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ has a n.h.s.,* *and* b) *there exists a single, specific,* C_{00}-*(BCP)-contraction* $B_u \in (BD)(\mathcal{H})$ *with* $\sigma_e(B_u) = \mathbb{D}^-$ *such that for each* $\varepsilon > 0$, *the collections of hyperlattices* $\{\mathrm{Hlat}(B_u + K) : K \in \mathbb{K}(\mathcal{H}), ||K|| < \varepsilon\}$ *and* $\{\mathrm{Hlat}(T) : T \in \mathcal{L}(\mathcal{H}) \backslash ((A) \cup \mathbb{C}1_{\mathcal{H}})\}$ *are identical up to lattice isomorphism).*

Theorem 2.1 raises some interesting questions concerning additional reductions that might be made to the hyperinvariant subspace problem. Let us write $(\mathbf{N}+\mathbf{K})$ or $(\mathbf{N}+\mathbf{K})(\mathcal{H})$ for the class of all operators T in $\mathcal{L}(\mathcal{H})$ such that T can be written as $T = N + K$ where N is normal and $K \in \mathbf{K}(\mathcal{H})$. It is well-known that $(\mathbf{N}+\mathbf{K}) \subsetneq (QD)$. Thus, for example, one may ask:

Problem 2.2: Is every T in $C_{00} \cap (\mathrm{BCP}) \cap (QD)(\mathcal{H})$ satisfying

$$\sigma_p(T) \cup \sigma_p(T^*) = \varnothing \quad \text{and} \quad \sigma(T) = \sigma_e(T) = \mathbb{D}^-$$

ampliation quasisimilar to some operator in $(\mathbf{N}+\mathbf{K})(\mathcal{H})$?

There is some evidence that the answer to Problem 2.2 might be affirmative. For instance, the following less known result from [1] shows that at least every operator in $\mathcal{L}(\mathcal{H})$ is a quasiaffine transform of some operator in $(\mathbf{N}+\mathbf{K})(\mathcal{H})$.

Theorem 2.3: *Suppose* $T \in \mathcal{L}(\mathcal{H})$ *and* $\varepsilon > 0$. *Then there exist* N, K, X *in* $\mathcal{L}(\mathcal{H})$ *such that* N *is normal,* $K \in \mathbf{K}(\mathcal{H})$, $\|K\| < \varepsilon$, X *is a quasiaffinity, and* $XT = (N + K)X$. *Moreover, if* T *is an algebraic operator, then* $T \sim N + K$.

Our first new result also points in this direction.

Theorem 2.4: *Every* $T \in (BD)(\mathcal{H})$ *is hyperquasisimilar to an operator in* $(\mathbf{N}+\mathbf{K})$. *Consequently, the set of all operators* T *in* (QD) *such that* T *is hyperquasisimilar to some element of* $(\mathbf{N}+\mathbf{K})$ *is norm-dense in* (QD).

Proof. Fix an arbitrary $T \in (BD)(\mathcal{H})$. By definition there exists a sequence $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ of orthogonal, reducing, finite dimensional subspaces of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ and $T = \bigoplus_{n\in\mathbb{N}} T_n$, where $T_n := T|_{\mathcal{H}_n}$. For each $n \in \mathbb{N}$ choose an orthonormal basis $\mathcal{E}_n$ for $\mathcal{H}_n$, and note that there exists an invertible $S_n \in \mathcal{L}(\mathcal{H}_n)$ such that the matrix $M_{\mathcal{E}_n}(S_nT_nS_n^{-1})$ of $S_nT_nS_n^{-1}$ with respect to the basis $\mathcal{E}_n$ is in Jordan canonical form, thus each $S_nT_nS_n^{-1}$ is the (orthogonal) direct sum $S_nT_nS_n^{-1} = \bigoplus_{k\in K_n} J_{k,n}$ of single Jordan blocks $J_{k,n}$, where each K_n is a finite index set. Moreover, it is clear (since each $\mathcal{E}_n$ is an orthonormal basis) that one has

$$\|S_n T_n S_n^{-1}\| \le r(T) + 1, \quad n \in \mathbb{N},$$

where, $r(T)$ denotes the spectral radius of T. Thus $\widetilde{T} = \bigoplus_{n\in\mathbb{N}} S_n T_n S_n^{-1} \in \mathcal{L}(\mathcal{H})$, T is hyperquasisimilar to $\widetilde{T}$, and thus it is suffices to show that $\widetilde{T}$ is hyperquasisimilar to an operator in $(\mathbf{N}+\mathbf{K})$. But it is obvious that $\widetilde{T}$ is the orthogonal direct sum $\bigoplus_n \left(\bigoplus_{k\in K_n} J_{k,n}\right)$, where each $J_{k,n}$ has a matrix (with respect to a suitable orthonormal basis $\mathcal{E}_{k,n} \subset \mathcal{E}_n$) that is a single Jordan block. Now let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers in the interval $(0,1)$ converging to 0, and note that it is an easy exercise in linear algebra that each Jordan block operator $J_{k,n}$, with, say,

$$M_{\mathcal{E}_{k,n}}(J_{k,n}) = \begin{pmatrix} \lambda_{k,n} & 1 & & & \\ & \lambda_{k,n} & 1 & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda_{k,n} \end{pmatrix},$$

is similar to an operator $\widehat{J}_{k,n}$ whose matrix relative to the same orthonormal basis is

$$\begin{pmatrix} \lambda_{k,n} & \varepsilon_n & & & \\ & \lambda_{k,n} & \varepsilon_n & & \\ & & \cdot & \cdot & \\ & & & \cdot & \varepsilon_n \\ & & & & \lambda_{k,n} \end{pmatrix}$$

(where the above diagonal 1's are replaced by ε_n's). Thus, $\widetilde{T}$ is hyperquasisimilar to $\widehat{T} = \bigoplus_{n\in\mathbb{N}} \left(\bigoplus_{k\in K_n} \widehat{J}_{k,n}\right)$, which also clearly satisfies $\|\widehat{T}\| \le r(\widetilde{T}) + 1$. Finally, it is obvious that $\widehat{T} \in (\mathbf{N}+\mathbf{K})(\mathcal{H}) \cap (BD)(\mathcal{H})$, since $\widehat{T}$ is the sum of a (diagonal) normal operator and the (countably infinite) orthogonal direct sum of a sequence of finite-rank operators with norms tending to zero. ∎

Unfortunately (for those interested in solving the hyperinvariant subspace problem affirmatively), despite these two positive results, we establish below that the answer to Problem 2.2 is negative, namely,

THEOREM 2.5: *There exists a (quasiaffinity)T in $C_{00} \cap (BCP) \cap (QD)$ satisfying $\sigma_p(T) \cup \sigma_p(T^*) = \varnothing$ and $\sigma_e(T) = \sigma(T) = \mathbb{D}^-$ such that T is ampliation quasisimilar to no operator in $(\mathbf{N}+\mathbf{K})$.*

3. The negative result

In this section we establish some results needed to prove Theorem 2.5. The first observation, which extends a result from [10], is as follows.

Proposition 3.1: *Suppose $S, T \in \mathcal{L}(\mathcal{H})$, $S \overset{a}{\sim} T$, and there exists a nonzero compact operator in $\{S\}'$. Then there exists a nonzero compact operator in $\{T\}'$. (In other words, the property of commuting with a nonzero compact operator is preserved by the relation $\overset{a}{\sim}$.)*

Proof. As mentioned above, $S \overset{a}{\sim} T$ implies $S^{(\aleph_0)} \sim T^{(\aleph_0)}$, so there exist quasiaffinities X, Y in $\mathcal{L}(\mathcal{H}^{(\aleph_0)})$ such that $XS^{(\aleph_0)} = T^{(\aleph_0)}X$ and $S^{(\aleph_0)}Y = YT^{(\aleph_0)}$. With $0 \neq K \in \{S\}' \cap \mathbf{K}(\mathcal{H})$, define $\widehat{K} \in \mathcal{L}(\mathcal{H}^{(\aleph_0)})$ by $\widehat{K} = K \bigoplus 0 \bigoplus \cdots$. Clearly $\widehat{K} \in \mathbf{K}(\mathcal{H}^{(\aleph_0)})$ and $\widehat{K}S^{(\aleph_0)} = S^{(\aleph_0)}\widehat{K}$. Moreover, $0 \neq X\widehat{K}Y \in \mathbf{K}(\mathcal{H}^{(\aleph_0)})$ and

$$T^{(\aleph_0)}(X\widehat{K}Y) = XS^{(\aleph_0)}\widehat{K}Y = X\widehat{K}S^{(\aleph_0)}Y = (X\widehat{K}Y)T^{(\aleph_0)},$$

so $T^{(\aleph_0)}$ commutes with the nonzero compact operator $X\widehat{K}Y$. Upon writing $X\widehat{K}Y$ as a $\aleph_0 \times \aleph_0$ matrix (K_{ij}) with operator entries from $\mathcal{L}(\mathcal{H})$ relative to the same direct sum decomposition $\mathcal{H}^{(\aleph_0)} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}$ that makes $T^{(\aleph_0)} = \mathrm{Diag}(T, T, \ldots)$, we get that each $K_{ij} \in \{T\}'$. Since $X\widehat{K}Y \neq 0$, some $K_{ij} \neq 0$, and the proof is complete. ∎

Corollary 3.2: *If $T \in \mathcal{L}(\mathcal{H})$, $0 \neq K \in \mathbf{K}(\mathcal{H})$, and $T \overset{a}{\sim} K$, then T commutes with a nonzero compact operator.*

Our next result improves both the statement and the proof of [10, Theorem 5].

Proposition 3.3: *There exists a quasinilpotent quasiaffinity $Q \in (QD)(\mathcal{H})$ such that $\{Q\}' \cap \mathbf{K} = \{0\}$ and thus $\sigma_p(Q) \cup \sigma_p(Q^*) = \varnothing$ and Q is not ampliation quasisimilar to any compact operator).*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be a fixed orthonormal basis for $\mathcal{H}$, and let $w = \{w_n\}_{n\in\mathbb{N}}$ be any bounded sequence of nonnegative numbers. We denote by U_w the (forward) weighted shift in $\mathcal{L}(\mathcal{H})$ defined by setting $U_w e_n = w_n e_{n+1}$ for $n \in \mathbb{N}$. By virtue of Corollary 3.2 it suffices to exhibit a quasinilpotent quasiaffinity Q in $(QD)(\mathcal{H})$ such that $\{Q\}' \cap \mathbf{K}(\mathcal{H}) = (0)$. For each $n \in \mathbb{N}$ let k_n be the largest nonnegative integer such that $2^{k_n - 1}$ is a factor of n. Thus, for example,

$$k_1 = k_3 = k_5 = \cdots = 1,\ k_2 = k_6 = k_{10} = \cdots = 2,\ k_4 = k_{12} = \cdots = 3, \text{ etc.}$$

Next define $w_n = 1/2^{(2^{k_n})}$, and consider U_w. Since examination shows that the largest product of m consecutive weights w_j is $\prod_{j=1}^{m} w_j$, it is clear that for every $m \in \mathbb{N}$, $\|(U_w)^m\| = \prod_{j=1}^{m} w_j$. Since the sequence $\{\|(U_w)^m\|^{1/m}\}$ converges to the spectral radius $r(U_w)$, to show that $r(U_w) = 0$ it suffices to exhibit a subsequence $\{m_n\} \subset \mathbb{N}$ such that $(\prod_{j=1}^{m_n} w_j)^{1/m_n} \to 0$. Moreover, if we set $m_n = 2^n - 1$ for $n \in \mathbb{N}$, then an easy calculation shows that

$$\sum_{j=1}^{2^n-1} 2^{k_j - n} = n, \quad n \in \mathbb{N},$$

and thus that

$$\begin{aligned}\lim_{n\to\infty}\left(\prod_{j=1}^{m_n} w_j\right)^{\frac{1}{m_n}} &= \lim_{n\to\infty}\left(\prod_{j=1}^{2^n-1}\frac{1}{2^{(2^{k_j})}}\right)^{\frac{1}{2^n-1}} = \lim_{n\to\infty} 2^{\frac{2^n}{2^n-1}(-\sum_{j=1}^{2^n} 2^{k_j-n})}\\ &= \lim_{n\to\infty} 2^{-\frac{n2^n}{2^n-1}}\\ &= 0,\end{aligned}$$

which shows that U_w is quasinilpotent. Since replacing any subsequence of weights $\{w_{n_j}\} \to 0$ in w by zero weights yields a block-diagonal operator, it is obvious that $U_w \in (BD) + \mathbf{K} = (QD)$. Moreover, by construction, no power $(U_w)^m$ with $m \in \mathbb{N}$ is compact. Since for $m_1 \neq m_2$, the nonzero entries of the matrices $M_{\mathcal{E}}(U_w^{m_1})$ and $M_{\mathcal{E}}(U_w^{m_2})$ lie on different diagonals, no formal power series in U_w (i.e., series of the form $\sum_{m=0}^{\infty} \alpha_m (U_w)^m$) is a nonzero compact operator, and it is well-known from [14], that $\{U_w\}'$ consists of formal power series in U_w. Now define

$$Q = \begin{pmatrix} U_w & e_1 \otimes e_1 \\ 0 & U_w^* \end{pmatrix} \in \mathcal{L}(\mathcal{H} \bigoplus \mathcal{H}) \tag{1}$$

(where $(e_1 \otimes e_1)x = \langle x, e_1 \rangle e_1$ for $x \in \mathcal{H}$), which is clearly a quasinilpotent bilateral weighted shift all of whose weights are nonzero. It follows trivially that Q is a quasiaffinity satisfying $\sigma_p(Q) \cup \sigma_p(Q^*) = \varnothing$. It is known that $\{Q\}'$ also consists of formal power series in Q [14], and since the $(1,1)$ entry of a formal power series in Q is a formal power series in U_w, it is easy to see that $\{Q\}' \cap \mathbf{K}(\mathcal{H}) = (0)$. Finally, since $U_w \bigoplus U_w^* \in (QD)$ and $e_1 \otimes e_1$ has rank one, clearly $Q \in (QD)$. ∎

Remark 3.4: The operator Q in (1) can clearly be written as $Q = B + K_1$, where $B \in (BD)$ and K_1 is a trace-class operator. This shows that restricting the compact perturbation of a block diagonal operator to be in some norm-ideal does not avoid the conclusion of Proposition 3.3.

PROPOSITION 3.5: *There exists a (quasiaffinity)*$T \in C_{00} \cap (BCP) \cap (QD)$ *satisfying* $\sigma_p(T) \cup \sigma_p(T^*) = \varnothing$ *and* $\sigma_e(T) = \sigma(T) = \mathbb{D}^-$ *such that* $\{T\}' \cap \mathbf{K} = (0)$.

Proof. Let Q be the quasinilpotent quasiaffinity in $\mathcal{L}(\mathcal{H})$ constructed in Proposition 3.3 such that $\{Q\}' \cap \mathbf{K} = (0)$ and normalized so that $\|Q\| = 1$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of distinct rational points in $\mathbb{D}$, and for each n in $\mathbb{N}$ choose $\alpha_n > 0$ such that $|\lambda_n| + \alpha_n < 1$. Define

$$Q_n = \alpha_n Q + \lambda_n 1_{\mathcal{H}}, \quad n \in \mathbb{N}, \tag{2}$$

and

$$T = \bigoplus_{n\in\mathbb{N}} Q_n, \tag{3}$$

relative to the usual decomposition $\mathcal{H}^{(\aleph_0)} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}$. Since $\|Q_n\| < 1$ and $\sigma_e(Q_n) = \sigma(Q_n) = \{\lambda_n\}$ for $n \in \mathbb{N}$, it is clear that T is a C_{00}-contraction and that $\sigma_{le}(T) = \sigma(T) = \mathbb{D}^-$. Thus $T \in (BCP)$, and since a (countable) direct sum of operators each of which belongs to (QD) and is a quasiaffinity also belongs to (QD) and is a quasiaffinity, we have that T is a quasiaffinity in (QD). Moreover, since $\sigma_p(Q) \cup \sigma_p(Q^*) = \varnothing$ it is trivial that $\sigma_p(T) \cup \sigma_p(T^*) = \varnothing$ also. Thus, it only remains to show that $\{T\}' \cap \mathbf{K}(\mathcal{H}^{(\aleph_0)}) = (0)$. Thus, suppose that $K \in \{T\}' \cap \mathbf{K}(\mathcal{H}^{(\aleph_0)})$, write $K = (K_{ij})_{i,j\in\mathbb{N}}$ as an $\aleph_0 \times \aleph_0$ matrix corresponding to the same decomposition as above, where (obviously) $K_{ij} \in \mathbf{K}(\mathcal{H})$ for all $i, j \in \mathbb{N}$. Since T may be written as $\mathrm{Diag}(Q_j)$, the matricial calculation

$$KT = TK = (K_{ij})(\mathrm{Diag}(Q_j)) = (\mathrm{Diag}(Q_i))(K_{ij})$$

gives

$$K_{ij} Q_j = Q_i K_{ij}, \quad i, j \in \mathbb{N}. \tag{4}$$

But since $\sigma(Q_j) = \{\lambda_j\}$ for $j \in \mathbb{N}$, and for $i \neq j$, $\lambda_i \neq \lambda_j$, by the Lumer–Rosenblum theorem, $K_{ij} = 0$ for all $i \neq j$. Moreover, for $i = j$, we get from (4) that $K_{jj} \in \{Q_j\}'$, and since $\{Q_j\}' = \{Q\}'$, we have that $K_{jj} = 0$ for $j \in \mathbb{N}$. ∎

Our next result is an improvement of [1, Theorem 4.8] and may be useful in showing that quasinilpotent operators have a n.h.s.

THEOREM 3.6: *If T is any operator (quasiaffinity) satisfying $\sigma_p(T)\cup\sigma_p(T^*) = \varnothing$ in $\mathcal{L}(\mathcal{H})$, $\sigma_e(T)$ is a singleton $\{\lambda_0\}$, and there exist R, X, $Y \in \mathcal{L}(\mathcal{H})$ such that $R \in (\mathbf{N}+\mathbf{K})(\mathcal{H})$, $TX = XR$, $YT = RY$, and $XRY \neq 0$, then T commutes with the nonzero compact operator $X(R-\lambda_0 1_{\mathcal{H}})Y$ and thus has a n.h.s.*

Proof. The proof splits into two cases: $\lambda_0 \neq 0$ and $\lambda_0 = 0$, so we suppose first that $\lambda_0 \neq 0$. It follows trivially from the hypotheses that XY, $XRY \in \{T\}'$ and $TXY = XRY \neq 0$. Thus $X \neq 0 \neq Y$. We now define $T_1 = T - \lambda_0 1_{\mathcal{H}}$ and $R_1 = R - \lambda_0 1_{\mathcal{H}}$. An easy computation gives that $T_1 X = XR_1$ and $YT_1 = R_1 Y$. We consider the possibility that $XR_1Y = 0$. But this would give

$$TXY = XRY = \lambda_0 XY \neq 0,$$

so trivially XY would be nonzero and λ_0 would belong to $\sigma_p(T)$, a contradiction. Thus $XR_1Y \neq 0$, and we note that since $\sigma_p(T_1)\cup\sigma_p(T_1^*) = \varnothing$, T_1 is a quasiaffinity, and T_1 bears the same relation to R_1 as T does to R (but with $\sigma_e(T_1) = \{0\}$). In other words, by the change of notation $T_1 \to T$, $R_1 \to R$, we have reduced the original problem to the case that $\sigma_e(T) = \{\lambda_0\} = \{0\}$. The remainder of the proof of this case is the same as the proof of [1, Theorem 4.8], but since it is short, we give it for ease of reference. Since $XRY \in \{T\}'$, it suffices to show that $XR \in \mathbf{K}$. Writing $R = N + K$ with N normal and $K \in \mathbf{K}$, we see that it is enough to show that $XN \in \mathbf{K}$. Moreover, writing $N = \int \lambda dE$ (so E is the spectral measure of N), and defining $E_n := E(\{\zeta \in \mathbb{C} : |\zeta| \geq 1/n\})$ for $n \in \mathbb{N}$, we have

$$\|XN - XE_nN\| \leq \|X\|\,\|N - E_nN\| \longrightarrow 0.$$

Thus it suffices to show that $XE_nN = XE_nNE_n \in \mathbf{K}$ for each $n \in \mathbb{N}$. Furthermore

$$TXE_n = X(N+K)E_n = XE_n(E_nNE_n + 1 - E_n) + XKE_n, \quad n \in \mathbb{N},$$

which gives, upon passing to the Calkin algebra,

$$\pi(T)\pi(XE_n) = \pi(XE_n)\pi(E_nNE_n) = \pi(XE_n)\{\pi(E_nNE_n + 1 - E_n)\},$$

and since $\sigma_e(T) = \sigma(\pi(T)) = \{0\}$ and $\sigma_e(E_nNE_n + 1 - E_n) \subset \sigma(E_nNE_n + (1 - E_n)) \subset \{\zeta \in \mathbb{C} : |\zeta| \geq 1/n\} \cup \{1\}$, the Lumer–Rosenblum theorem (applied in the Calkin algebra) gives that $XE_n \in \mathbf{K}$ for $n \in \mathbb{N}$, which completes the proof. ∎

Proposition 3.7: *Let $T \in \mathcal{L}(\mathcal{H}^{(\aleph_0)})$ be the quasiaffinity constructed in Proposition 3.5. Then there exists no triple (R, X, Y) having all four of the following properties:*

A) $R \in (\mathbf{N}+\mathbf{K})(\mathcal{H})$, $X, Y \in \mathcal{L}(\mathcal{H}^{(\aleph_0)})$,
B) $TX = XR^{(\aleph_0)}$,
C) $YT = R^{(\aleph_0)}Y$,
D) $XR^{(\aleph_0)}Y \neq 0$.

Proof. Suppose such a triple (R, X, Y) exists with $R^{(\aleph_0)}$ the diagonal matrix $\mathrm{Diag}(R, R, \ldots)$ relative to the usual decomposition $\mathcal{H}^{(\aleph_0)} = \mathcal{H} \bigoplus \mathcal{H} \bigoplus \cdots$, and write X and Y as $\aleph_0 \times \aleph_0$ matrices $X = (X_{ij})$, $Y = (Y_{ij})$, with entries $X_{ij}, Y_{ij} \in \mathcal{L}(\mathcal{H})$ corresponding to the same decomposition. Then B) and C) give immediately that

$$(5) \qquad Q_i X_{ij} = X_{ij} R, \quad Y_{ij} Q_j = R Y_{ij}, \quad i, j \in \mathbb{N},$$

and therefore

$$(6) \qquad (X_{ij} R Y_{jk}) Q_k = Q_i (X_{ij} R Y_{jk}), \quad i, j, k \in \mathbb{N}.$$

For all i, $k \in \mathbb{N}$ with $i \neq k$, by the Lumer–Rosenblum theorem (since $\sigma(Q_k) = \{\lambda_k\} \neq \{\lambda_i\} = \sigma(Q_i)$),

$$X_{ij} R Y_{jk} = 0, \quad i, j, k \in \mathbb{N}, i \neq k.$$

Hence, $XR^{(\aleph_0)}Y = (X_{ij}) \,\mathrm{Diag}(R, R, \ldots)(Y_{ij}) := (Z_{ij}) = (\sum_k X_{ik} R Y_{kj})$ satisfies $Z_{ij} = 0$ whenever $i \neq j$, and applying D), we obtain that some $Z_{kk} (= \sum_j X_{kj} R Y_{jk}) \neq 0$, and consequently for some $j \in \mathbb{N}$, $X_{kj} R Y_{jk} \neq 0$. Since $R \in (\mathbf{N}+\mathbf{K})$, $\sigma(Q_k) = \{\lambda_k\}$, and (5) is valid, we obtain immediately from Theorem 3.6 that $Q_k = \alpha_k Q + \lambda_k 1_{\mathcal{H}}$ commutes with the nonzero compact operator $X_{kj} R Y_{jk}$ which contradicts the fact that $\{Q\}' \cap \mathbf{K} = (0)$ (via Proposition 3.3). ∎

On the basis of the preceding results in this section, we are now prepared to prove Theorem 2.5.

Proof of Theorem 2.5. Let T be the quasiaffinity constructed in Proposition 3.5 (and defined by (3)). Suppose now that there exists $R \in (\mathbf{N}+\mathbf{K})$ such that T is ampliation quasisimilar to R, i.e., $T^{(\aleph_0)} \sim R^{(\aleph_0)}$. Since $R^{(\aleph_0)}$ is unitarily equivalent to $(R^{(\aleph_0)})^{(\aleph_0)}$ we have $T^{(\aleph_0)} \sim (R^{(\aleph_0)})^{(\aleph_0)}$, and, relative to the usual decomposition $\mathcal{K}^{(\aleph_0)} = \bigoplus_{n\in\mathbb{N}} \mathcal{K}$ with $\mathcal{K} = \mathcal{H}^{(\aleph_0)}$ for $n \in \mathbb{N}$, $T^{(\aleph_0)}$ becomes the diagonal matrix $\mathrm{Diag}(T, T, \ldots)$. Upon application of another unitary equivalence to $(R^{(\aleph_0)})^{(\aleph_0)}$, followed by a change of notation, we may also suppose that $(R^{(\aleph_0)})^{(\aleph_0)}$ is the diagonal matrix $\mathrm{Diag}(R^{(\aleph_0)}, R^{(\aleph_0)}, \ldots)$, where $R \in \mathcal{L}(\mathcal{H})$. Moreover there exist quasiaffinities X, $Y \in \mathcal{L}(\mathcal{K}^{(\aleph_0)})$ such that $T^{(\aleph_0)}X = X(R^{(\aleph_0)})^{(\aleph_0)}$ and $YT^{(\aleph_0)} = (R^{(\aleph_0)})^{(\aleph_0)}Y$. If we write X and Y as $\aleph_0 \times \aleph_0$ matrices $X = (X_{ij})$, $Y = (Y_{ij})$, with entries $X_{ij}, Y_{ij} \in \mathcal{L}(\mathcal{H}^{(\aleph_0)})$ relative to this same decomposition $\mathcal{K}^{(\aleph_0)} = \bigoplus_{n\in\mathbb{N}} \mathcal{K}$, we obtain

$$TX_{ij} = X_{ij}R^{(\aleph_0)}, \quad Y_{ij}T = R^{(\aleph_0)}Y_{ij}, \quad i, j \in \mathbb{N}.$$

Since $T \neq 0$, we have $R \neq 0$ and $X(R^{(\aleph_0)})^{(\aleph_0)}Y \neq 0$. Thus, by an argument similar to that in the proof of Proposition 3.7, there exist i_0, j_0, k_0 in $\mathbb{N}$ such that

$$X_{i_0j_0}R^{(\aleph_0)}Y_{j_0k_0} \neq 0.$$

But the existence of the triple $(R, X_{i_0j_0}, Y_{j_0k_0})$ contradicts Proposition 3.7, and the proof is complete. ∎

Remark 3.8: Of course, it is well-known (cf. [14]) that every quasinilpotent quasiaffinity Q that is a weighted bilateral shift has the property that $\{Q\}'$ consists exactly of all formal power series in Q and thus has a good supply of n.h.s. Moreover, one knows from [8] that if at least one direct summand in any (countable) direct sum of operators has a n.h.s., then so does the direct sum. Consequently the quasiaffinity T of Theorem 2.5 (and defined in Proposition 3.5) does have a n.h.s.

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